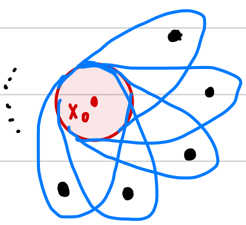
Math 451: Introduction to General Topology
Lecture 15

The flower topology. We now give an example of topological space which shows
$$2^{\text{nd}} \text{ ctbl} \;\cancel{\Leftarrow}\; 1^{\text{st}} \text{ ctbl} + \text{separable}.$$

Let X be a set (e.g. $X := \mathbb{R}$) and $x_0 \in X$ (e.g. $x_0 := 0 \in \mathbb{R}$). Let $\mathcal{B} := \{\{x_0, x\} : x \in X\}$; in particular, $\{x_0\} = \{x_0, x_0\} \in \mathcal{B}$. Call the topology on X generated by $\mathcal{B}$ the *flower topology on X with center x_0*. Since $\mathcal{B}$ is closed under finite intersections, $\mathcal{B}$ is a basis for this topology. This topology is:



- 1^{st} ctbl because for each $x \in X$, the set $\{\{x_0, x\}\}$ is a neighbourhood basis at x.
- separable because $\{x_0\}$ is dense in X since x_0 is contained in every open set.
- is 2^{nd} ctbl $\Leftrightarrow$ X is ctbl.

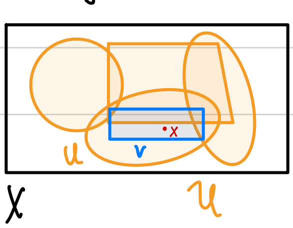
  Proof. $\Leftarrow$. If X is ctbl then $\mathcal{B}$ is ctbl and is a basis.

  $\Rightarrow$. We prove the contrapositive: assume X is unctbl and let $\mathcal{C}$ be any basis for the topology. We show that $\mathcal{B} \subseteq \mathcal{C}$, hence $\mathcal{C}$ is unctbl because $\mathcal{B} \cong X$.
  Fix a $P := \{x_0, x\} \in \mathcal{B}$. Since $\mathcal{C}$ is a basis, P is a union of sets in $\mathcal{C}$. But the only open subsets of P are $\emptyset$, $\{x_0\}$, and P itself, so if P is a union of open sets, one of them must be P to cover the point x, hence $P \in \mathcal{C}$. $\square$

What we really proved is that $\mathcal{B}$ is contained in every basis of X. Thus, the flower top. on $\mathbb{R}$ with any center is 1^{st} ctbl, separable, but not 2^{nd} ctbl.

Prop (AC). 2^{nd} ctbl $\Rightarrow$ Lindelöf.

Proof. Let $\mathcal{B}$ be a ctbl basis for the top. space X, let $\mathcal{U}$ be an open cover of X, and our goal is to find a ctbl subcover. Let $\mathcal{B}_0 := \{V \in \mathcal{B} : \exists U \in \mathcal{U} \text{ s.t. } V \subseteq U\} \subseteq \mathcal{B}$ hence still ctbl.



By AC, define a function $\mathcal{B}_0 \to \mathcal{U} : V \mapsto U_V \supseteq V$. Now let $\mathcal{U}_0 := \{U_V : V \in \mathcal{B}_0\}$. $\mathcal{B}_0$ surjects onto $\mathcal{U}_0$, hence $\mathcal{U}_0$ is ctbl. Also, $\mathcal{U}_0$ is a cover of X because for each $x \in X$, $\exists\, U \in \mathcal{U}$ with $x \in U$, but then since $\mathcal{B}$ is basis, $\exists\, V \in \mathcal{B}$ s.t. $x \in V \subseteq U$. Thus, $V \in \mathcal{B}_0$, hence $U_V \in \mathcal{U}_0$ and $U_V \supseteq V \ni x$. $\square$

Cor. 2^{nd} ctbl $\Rightarrow$ 1^{st} ctbl + separable + Lindelöf.

The converse still fails:

Sorgenfrey line. The following space shows:

$$2^{nd} \text{ ctbl} \not\Leftarrow 1^{st} \text{ ctbl} + \text{separable} + \text{Lindelöf}.$$

Let $X := \mathbb{R}$ and let $\mathcal{S}$ be the topology on $\mathbb{R}$ generated by

$$\mathcal{B} := \{[a, b) : a, b \in \mathbb{R}, a \leq b\},$$

called the Sorgenfrey topology on $\mathbb{R}$.

Claim. This topology is finer than the usual top on $\mathbb{R}$, i.e. it has more open sets (formally, every usual open set is Sorgenfrey open).

Proof. Because open intervals (a, b), $a, b \in \mathbb{R}, a < b$ form a basis for the usual top. on $\mathbb{R}$, it is enough to show that (a, b) is Sorgenfrey open. But $(a, b) = \bigcup_{n \in \mathbb{N}^+} [a + \frac{1}{n}, b) = \bigcup_{a < a' < b} [a', b)$, so (a, b) is Sorgenfrey open. $\square$

Because $\mathcal{B}$ is closed under finite intersections, $\mathcal{B}$ is a basis for $\mathcal{S}$. It is left as a HW exercise to show that this topology is 1^{st} ctbl, separable, Lindelöf, but not 2^{nd} ctbl.

Recall that metric spaces are 1^{st} ctbl. It turns out the remaining three properties are equivalent to each other for metric spaces.

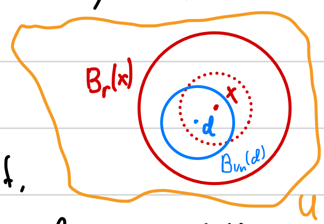
Theorem. For a metric space (X, d), TFAE:

(1) X is 2^{nd} ctbl.

(2) X is Lindelöf.

(3) X is separable.

Proof. (1)$\Rightarrow$(2). Already proved generally.
(2)$\Rightarrow$(3). HW.
(3)$\Rightarrow$(1). Let $D \subseteq X$ be a ctbl dense set. Let $\mathcal{B} := \{B_{1/n}(d) : d \in D, n \in \mathbb{N}^+\}$, so $\mathcal{B}$ is ctbl and we show it is a basis. Fix an open set U and we need to show that for each $x \in U$ $\exists$ $B \in \mathcal{B}$ s.t. $x \in B \subseteq U$. Fix $x \in U$. Since U is open, there is a ball $B_r(x) \subseteq U$ for some $r > 0$. Let $n \in \mathbb{N}^+$ be s.t. $\frac{1}{n} < \frac{r}{2}$. Then by density, $\exists$ $d \in D \cap B_{1/n}(x)$, so $B_{1/n}(d) \subseteq B_r(x) \subseteq U$ and $x \in B_{\frac{1}{n}}(d)$ since $d \in B_{1/n}(x)$. $\square$



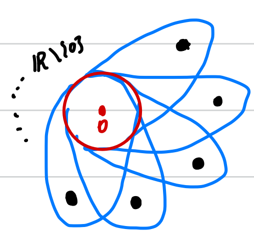
Hereditariness of countability properties.

We now let X be a top. space, $Y \subseteq X$ equipped with the subspace topology, and discuss which of the ctblity properties are inherited by Y from X.

Obs. Let X be a top. space and $Y \subseteq X$.
(a) If X is 1^{st} ctbl then Y is 1^{st} ctbl.
(b) If X is 2^{nd} ctbl then Y is 2^{nd} ctbl.

Obs. If X is Lindelöf and Y is closed, then Y is also Lindelöf.
Proof. HW.

Counterexample to inheritance of separability. Take the flower top on $\mathbb{R}$ with center 0. Then as shown above this top. is separable since $\{0\}$ is dense but $Y := \mathbb{R} \setminus \{0\}$ is not separable because the subspace top on Y is discrete! Indeed, all singletons in Y are open since for each $y \in Y$, $\{y\} = \{0, y\} \cap Y$. Thus, Y is not separable since it is unctbl and discrete.



Cor. For metric spaces, separability and Lindelöfness are hereditary, i.e. if X is metric space and $Y \subseteq X$, then X separable (resp. Lindelöf) $\Rightarrow$ Y is separable (resp. Lindelöf).

Proof. For metric spaces these properties are equivalent to 2^{nd} countability, which is always hereditary. $\square$

Convergence in topological spaces.

Def. Let X be top. space and $(x_n) \subseteq X$. We say that (x_n) converges to some $x \in X$, and write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$, if for each open neighbourhood $U \ni x$ all but finitely many $x_n \in U$, i.e. $\forall^\infty n \ \ x_n \in U$.

Prop. Let X be top. space, $x \in X$, and $\mathcal{B}_x$ a neighbourhood basis at x. Then for any sequence $(x_n) \subseteq X$, $x_n \to x$ as $n \to \infty$ $\Leftrightarrow$ for each $U \in \mathcal{B}_x$ $\forall^\infty_n \ x_n \in U$.

Proof. HW.